MODERN ANALYSIS

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Subspace

Let (M, d) be a metric space. Let M_1 be a non-empty subset off M. Then M_1 is also a metric space with the same metric d. We say that (M_1, d) is a subspace of (M, d).

Note : If M_1 is a subspace of M a set which is open in M_1 need not be open in M. For example, If $M = R$ with usual metric and $M_1 = [0, 1]$ then $(0, \frac{1}{2})$ is open in M_1 but not open in M.

Theorem

Let M be a metric space and M_1 a subspace of M. Let $A_1 \subseteq M_1$ iff there exists an open set A in M such that $A_1 = A \cap M_1$.

Example

Let $M = R$ and $M_1 = [0, 1]$. Let $A_1 = [0, \frac{1}{2})$.

Example

Let $M = R$ and $M_1 = [1, 2] \cup [3, 4]$.

Problems:

1. Let M_1 be a subspace of a metricspace M. Prove that every open set A_1 of M_1 is open in M iff M_1 itself is open in M.

2. Give an example of a metric space M and a non-empty proper subspace M_1 of M such that every open set in M_1 is also an open set in M.

3. Let M_1 be a subspace of a metric space M. Let $A_1 \subseteq M_1$. If A_1 is open in M prove that it is open in M_1. also.

Interior of a Set

Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an interior point of A if there exists a positive real number such that $B(x, r) \subseteq A$.
The set of all interior points of A is called the interior of A and it is donoted by *Int* A. Note that *Int* $A \subseteq A$.

Example

Consider R with usual metric.
(a) Let $A = [0, 1]$. Clearly 0 and 1 are nto interior points of A and any point $x \in (0, 1)$ is an interior point of A. Hence *Int* $A = (0, 1)$.
(b) Let $A = Q$. Let $x \in Q$. Then *Int* Q = φ.
(c) Let A be a finite subset of R. Then *Int* $A = \varphi$.

Examples

1) Consider R with discrete metric. Let $A = [0, 1]$. Here *Int* A = A.

2) In a discrete metric space M, *Int* $A = A$, for any subset A of M.

Theorem

(i) A is open iff $A = Int\ A$.
In Particular $Int\ \phi = \phi$ and $Int\ M = M$.
(ii) $Int\ A$ = Union of all open sets contained in A.
(iii) $Int\ A$ is open subset of A and if B is any other open set contained in A then $B \subseteq$IntA.
i.e. $Int\ A$ is the largest open set contained in A.
(iv)$A \subseteq B \Longrightarrow$IntA$\subseteq$IntB.
(v)$Int\ (A)$ = $Int\ A\ \cap$IntB.
(vi)$Int\ (A \cup B)\ \supseteq$IntA$\supseteq$IntB.

Definition

Let (M, d) be a metric space. Let $A \subseteq M$. Then A is said to be **closed** in M if the complement of A is open in M.

Example

1. In R with usual metric any closed interval $[a, b]$ is closed set.
2. In R with usual metric $[a, b)$ is neither closed nor open.
3. In R with usual metric $(a, b]$ is neither closed nor open.
4. Z is closed.
5. Q is closed in R.
6. The set of irrational numberrs is not closed in R.
7. In R with usual metric every singleton set is closed.
8. Every subset of a discrete metrice space is closed.

Closed Ball

Let (M, d) be a metric space. Let $a \in M$. Let r be any positive real number. Then closed ball or the closed sphere with centre a and radius r, denoted by $B_d[a, r]$, is defined by $B_d[a, r] = \{x \in M / d(a, x) \leq r\}$.

Examples

1. In R with usual metric $B[a, r] = [a - r, a + r]$.
2. In R^2 with usual metric let $a = (a_1, a_2) \in R^2$. Then $B[a, r]$ is the set of all points which lie within and on the circumference of the circle with centre a and radius r.

Theorems

1. In any metric space every closed ball is a closed set.
2. In any metric space M, (i) ϕ is closed (ii) M is closed.
3. In any metric space the union of a finite number of closed sets is closed.

Note: The union of an infinite collection of closed sets need not be closed. consider R with usual metric. Let $A_n = [\frac{1}{n}, 1]$ where n= 1,2,3...

Introduction

Theorem

Let M be a metric space and M_1 be a subspace of M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 iff there exists a set F which is closed in M such that $F_1 = F \cap M_1$.

Closure

Let A be a subset of metric space (M, d). The Closure of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A.

Thus $\bar{A} = \cup\{B/B$ is closed in M and $A \subseteq B\}$. Note that $\bar{A}$ is the smallest closed set containing A.

Theorem

1. A is closed iff $A = \bar{A}$.
2. $\phi = \bar{\phi}$
3. $M = \bar{M}$
4. $\bar{\bar{A}} = \bar{A}$

Theorem

In a discrete metric space (M, d) any subset A of M is closed. Hence $\bar{A} = A$.

Theorem

Let (M, d) be a metric space. Let $A, B \subseteq M$. Then
(i) $A \subseteq B \implies \implies \bar{B}$.
(ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
(iii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

THANK YOU